Jech the axiom of choice pdf

I'm not robot

reCAPTCHA

Continue

Comprehensive in its selection of topics and results, this stand-alone text examines the relative strengths and consequences of the axiom of choice. Each chapter contains several problems, classified according to difficulty, and ends with some historical observations. An introduction to the use of the axiom of choice is followed by explorations of consistency, patterns of permutation and independence. The following chapters examine incorporation theorems, finished-support models, weaker versions of the axiom, and non-transferable declarations. The final sections consider unelected math, cardinal numbers in the theory set without choice, and properties that contradict the axiom of choice, including the axiom of determinancy and related topics the Basic Committee Library List suggest that the licensee math libraries consider this book for purchase. MAA Review Contents Preface 1. Introduction 2. Use of the axiom of choice 3. Consistency of the axiom of choice 4. Permutation Models 5. Independence of the axiom of choice 6. Incorporation theorems 7. Models with finished support 8. Some weaker versions of the axiom of choice 9. Non-transferable declarations 10. Mathematics without choice 11. Cardinal numbers in set theory without choice 12. Some properties contradict the appendix 1 axiom of choice. Equivalents of the axiom chosen Appendix 2. Equivalents of the Theorem of the First Ideal Appendix 3. Various independence results Appendix 4. Various Examples Author Index Index Symbol List first published Mar 8 January 2008; background review Wednesday 18 March 2015 The principle of the theory of the set known as the Axiomofa of Choice was hailed as probably the most interesting and, despite its late appearance, the most discussed axiom of mathematics, the second after Euclid's parallel axiom, which was introduced more than two thousand years ago (Fraenkel , Bar-Hillel & Levy 1973, §II.4). The fulness of this description could lead those who are not familiar with the axiom to expect it to be as surprising as, say, the principle of constancy of the speed of light or the principle of Heisenberg uncertainty. But in fact, Axiom of choice, so it is usually said humdrum appears, even by itself obvious. Because it amounts to nothing more than the claim that, given any collection of mutually dissentless nonempty sets, it is possible to assemble a new set-a-cross set or choice-containing exactly one element from each member of the given collection. However, this seemingly innocuous principle has far-reaching mathematical consequences – many indispensable, some amazing – and has come to present itself visibly in discussions about the fundamentals of mathematics. It (or its equivalents) have been used in countless mathematical works, and a number of monographs have been dedicated exclusively to it. In 1904 Ernst Zermelo formulated Axioma (abbreviated as AC throughout this article) in terms of what he called coatings (Zermelo 1904). 1904). starts with an arbitrary set \(M\) and uses the symbol \(M'\) to designate an arbitrary nonempty subset of \(M\), the collection whose collection denotes M. He continues: Imagine that with each subset \(M'\) there is an arbitrary element \(m_{1}'\), which appears in \(M'\) itself; to \(m_{1}'\) to be called the distinguished element of \(M'\). This produces a \(g\) coverage of the \(M\) set of certain elements of the \(M\) set. The number of these coatings is equal to the product [cardinalities of all \(M'\) subsets] and is certainly different from 0. The last sentence of this quote — which states, in fact, that the coatings always exist for the collection of the unempty subsets of any set (neempty) — is the first formulation of the Axiom of Choice by Zermelo[1]. This is now usually stated in terms of choice functions: here a choice function on a \(\sH\) collection of nonempty sets is a map \(f\) with \(\sH\) domain so that \(f(X) \in X\) for each \(X \in \sH\). As a very simple example, let \(\sH\) be the collection of unempty subsets of \(\{0, 1\}\), ie, \(\sH = \{\{0\}, \{1\}, \{0,1\}\}\). Then \(\sH\) has the two distinct choice functions \(f_{1}\) and \(f_{2}\) given by: \begin{align} f_{1}(\{0\}) &= 0 \\ f_{1}(\{1\}) &= 1 \\ f_{1}(\{0), 1\}) &= 0 \\ f_{2}(\{0\}) &= 0 \\ f_{2}({1\}) &= 1 \\ f_{2}(\{0, 1\}) &= 1 \end{align} A more interesting example of a choice function is provided by taking \(\sH\) to be the set of pairs (unordered) of actual numbers and the function that assigns each pair at least element. A different choice function is achieved by assigning each pair its largest element. Clearly, many more choice functions can be defined on \(\sH\). Declared in terms of choice functions, Zermelo's first formulation by AC reads: AC1: Any collection of nonempty sets has a function of choice. AC1 can be reformulated in terms of indexed or variable sets. An indexed set collection \(\sA = \{A_{i}: i \in I\}\) can be designed as a set of variables, with that being, as a set that varies over the index set \(I\). Each \(A_{i}\) is then the value of the set of variables \(\sA\) in the \(i\) stage. A choice function in \(\sA\) is a map \(f: I \rightarrow \bigcup_{i\in I} A_{i}\) so that \(f(i) \in A_{i}\) for all \(i\in I\). A choice function on \(\sA\) is thus a choice of an element of the \(\sA\) variable set at each stage; in other words, a choice function in \(\sA\) is a variable element of \(\sA\). AC1 is then equivalent to the statement AC2: Any indexed collection of sets has a choice function. From an informal point of view, AC2 equates to the assertion that a set of variables with an element in each stage has a variable element. AC1 can also be reformulated in terms of relationships, viz. AC3: For any relationship \(R\) between sets \(B\), \[{\forall x\inn A}\ \exist y\inn B[R(x,y)] \Rightarrow \exists f[f: A \rightarrow B \amp {\forall x\inn A}[R(x,fx)]]. Finally it is easy to prove to be equivalent (in established ordinary theories) to:[2] AC4: Any surjective function has a reverse right. In a 1908 paper, Zermelo introduced a modified form of AC. Let's call a cross-sectional set (or choice) for a family of sets \(\sH\) any subset \(T \subseteq \bigcup \sH\) for which each intersection \(T \cap X\) for \(X \in \sH\) has exactly one element. As a very simple example, let \(\sH = \{\{0\}, \{1\}, \{2, 3\}\}\). Then \(\sH\) has the two transverse \(\{0, 1, 2\}\) and \(\{0, 1, 3\}\). A more substantial example is provided by allowing \(\sH\) to be the collection of all lines in the Euclidean plane parallel to the \(x\) axis. Then the \(T\) set of points on the \(y\)-axis is a transverse for \(\sH\). Declared in terms of transverses, then, The second formulation (1908) of The AC's Zermelo equates to the assertion that any family of non-mutual sets has a transverse. [3] Zermelo states that the purely objective nature of this principle is immediately obvious. In making this claim, Zermelo wanted to point out that, in this form, the principle makes no use to the possibility of making choices. Zermelo may also consider the following combination justification of the principle. Considering a family \(\sH\) of nonempty sets disarticulation to each other, call a subset \(S \subseteq \bigcup \sH\) a selector for \(\sH\) if \(S\cap X e \varnothinging\) for all \(X \in \sH\). There are clear selectors for \(\sH\); \(\bigcup \sH\) is an example. Now we can imagine taking a \(S\) selector for \(\sH\) and thinning each intersection \(S \cap X\) for \(X\in \sH\) until it contains only one item. The result is a crossfor \(\sH\). This argument, duly refined, produces a precise derivation of AC in this formulation of the set-theoretical principle known as Zorn (see below). Let's call Zermelo's 1908 combination axiom of choice: LAC: Any collection of mutually disjunction nonempty sets has a transverse. It should be noted that AC1 and for finished collections of sets are both demonstrable (by induction) in the theories of the ordinary set. But in the case of an infinite collection, even when each of its members is finite, the question of the existence of a function of choice or a transverse is problematic. For example, after already mentioned, it is easy to come up with a choice function for the collection of pairs of real numbers (simply choose the smaller item of each pair). But it is not at all obvious to produce a function of choice for collecting pairs of arbitrary sets of real numbers. Zermelo's original purpose in introducing AC was to establish a central principle of Cantor's set theory, namely, that each set admits a good order and thus can also be attributed a number Zermelo's introduction of the axiom in 1904, as well as the use it used, provoked considerable criticism from mathematicians at the time. The main objection raised was to some have seen it as extremely unconstructive, even idealistic: while the axiom affirms the possibility of making a number of - perhaps even an innumerable number - of arbitrary choices, it does not provide any indication of how the latter are actually to be performed, by how otherwise put, the functions of choice must be defined. This was particularly unacceptable to the mathematicians of a constructive bent, would be the so-called French empiricists Baire, Borel and Lebesgue, for whom a mathematical object could be asserted to exist only if it could be defined in such a way as to characterise it in a unique way. Zermelo's response to his critics came in two works in 1908. In the first of these, after mentioned above, he reformulated the AC in terms of transverses; in the second (1908a) he made explicit the additional hypotheses necessary to carry out the proof of the theorem of ordering. These hypotheses were the first explicit presentation of an axiom system for set theory. As the debate over the Axiom of Choice began to last, it became apparent that the evidence of a number of significant mathematical theorems made essential use of it, thus leading many mathematicians to treat it as an indispensable tool of their trade. Hilbert, for example, came to regard AC as an essential principle of mathematics[5] and used it in defense of classical mathematical reasoning against attacks by intuitionists. Indeed, its ε operators are essentially only functions of choice (see entry on the epsilon calculation). Although the usefulness of the AC is rapidly becoming clear, doubts about its solidity remain. These doubts were reinforced by the fact that they had certain striking counterintuitive consequences. The most spectacular of these was the paradoxical decomposition of the sphere by Banach and Tarski (Banach and Tarski 1924): any solid sphere can be divided into several pieces that can be reassembled to form two solid spheres of the same size; and any solid sphere can be divided into infinitely several pieces, so as to allow them to be reassembled to form a solid sphere of arbitrary dimensions. (See Car 1993.) It was not until the mid-1930s that the question of AC solidity was finally worked out with Kurt Gödel's proof of coherence with the other axioms of set theory. Here is a brief timeline of AC:[6] 1904/1908 Zermelo introduces axioms of the theory of the set, explicitly formulates the AC and uses it to prove the well-ordered theorem, thus raising a storm of controversy. 1904 Russell recognizes AC as a multiplier axiom: the product of arbitrary nezero cardinal numbers is non-zero. 1914 Hausdorff derives from the AC the existence of immeasurable sets in the paradoxical form that \(\bfrac{1}{2}\) of a sphere is congruent with the \(\bfrac{1}{3}\) of it (Hausdorff 1914). Fraenkel introduces the permutation method to establish AC independence from an atom set theory system (Fraenkel 1922). 1924 Building on by Hausdorff, Banach and Tarski derive from AC their paradoxical decompositions of the sphere. 1926 Hilbert introduces in his theory transfinite proof or axiom epsilon as a version of AC . (Hilbert 1926). 1936 Blendbaum and Mostowski expand and refine Fraenkel's method of permutation and prove the independence of the various statements of the set theory weaker than AC. (Lindenbaum and Tarski 1938) 1935–38 Gödel establishes the relative consistency of AC with the axioms of set theory (Gödel 1938a, 1938b, 1939, 1940). 1950 Mendelson, Shoenfield and Specker, working independently, use the permutation method to establish the independence of different forms of AC from a system of set theory without atoms, but also devoid of foundation axiom (Mendelson 1956, 1958, Shoenfield 1955, Specker 1957). 1963 Paul Cohen proves THE independence of the AC from the standard axioms of the theory of the set (Cohen 1963, 1963a, 1964). 2. The independence and coherence of the Axiom of Choice After mentioned above, in 1922 Fraenkel proved the independence of the AC from a system of theory of the set containing atoms. Here of an atom is understood a pure individual, that is, an entity that does not have members and still distinct from the empty set (so a fortiori an atom cannot be a set). In a system of the theory of the set with atoms one is assumed to be given an infinite set \(A\) of atoms. You can build a \(V(A)\) universe of sets over \(A\) starting with \(A\), adding all subsets of \(A\), adjacent to all subsets of the result, etc., and itering transfinitely. \(V(A)\) is then a model of the theory of the set of atoms. The core of Fraenkel's method of proving AC independence is the observation that, since atoms cannot be theoretically differentiated, any permutation of the set \(A\) of atoms induces a permutation of structural preservation — an automorphism — of the \(V(A)\) universe of sets constructed from \(A\). This idea can be used to build another \(Sym(V)\) model of the set theory — a permutation or a symmetrical pattern — in which a set of pairs of \(A\) elements has no choice function. Now, suppose we are given a group \(G\) of automorphisms of \(A\). Let's say an automorphism \(\pi\); \(A\) fixes an \(x\) item in \(V(A)\) if \(\pi(x) = x\). Clearly, if \(\pi \in G\) repairs each element of \(A\), fix each element of \(V(A)\). Now, for certain \(x \in V(A)\items, the attachment of elements to a subset of \(A\) by any \(\pi\in G\) is sufficient to fix \(x\). Therefore, we are led to define a support for \(x\) to be a subset of \(X\) of \(A\) so that whenever \(\pi\in G\) determines each member of \(X\), it also fixes \(x\). Members \(V(A)\) that possess a finite support are called symmetrical. Next we define the universe \(Sym(V)\) to consist of hereditary symmetric members of \(V(A)\), i.e. those \(x\in V(A)\) so that \(x\), \(x\), the elements of \(x\), etc., are all symmetrical. \(Sym(V)\) is also a set model with a set of atoms \(A\), and \(\pi\) induces an automorphism of \(Sym(V)\). Now suppose that \(A\) is partitioned into a (necessarily infinite) mutual disjunct set \(P\) pairs. Take \(G\) to be the permutation group of \(A\) that repairs all pairs in \(P\). Then \(P\in Sym(V)\); it can now be shown that \(Sym(V)\) does not contain any choice function on \(P\). To assume \(f\) were a choice function on \(P\) and \(f \in Sym(V)\). Then \(f\) has a finite support that can be considered to be of the form \(\{a_{1}, \ldots, a_{n}, b_{1},\ldots,b_{n}\} with each pair \{a_{i}, b_{i}\} \in P\). Because \(P\) is infinite, we can select a pair \(\{c, d\} = U\) from \(P\) different from all \(\{a_{i}, b_{i}\}\). We now define \(\pi \in G\) so that \(\pi\) corrects each \(a_{i}\) and \(b_{i}\) and interchanges \(c\) and \(d\). Then \(\pi\) fixes and \(f\). Because \(f\) was supposed to be a choice function on \(P\) and \(U \in P\), we must have \(f(U) \in U\), i.e. ,(f(U) = c\) or \(f(U) = d\). Because \(\pi\) exchanges \(c\) and \(d\), it follows that \(\pi(f(F)) is f(U)\). But because \(\pi\) is an automorphism, it also retains the application function, so that \(\pi(f(U)) = \pi f(\pi(U))\). But \(\pi(U) = U\) and \(\pi f = f\), from where \(\pi(f(U)) = f(U)\). We have duly come to a contradiction, which shows that the universe \(Sym(V)\) does not contain any choice function on \(P\). The idea here is that for a symmetrical function \(f\) defined on \(P\) there is a finite \(L\) list of pairs in \(P\) fixing all items whose elements are sufficient to fix \(f\), and therefore all \(f\) values. Now, for any pair \(U\) in \(P\) but not in \(L\) , you can always find a \(\pi\) permutation that fixes all pair elements in \(L\), but does not fix \(U\) members. Because \(\pi\) must set the value \(f\) to \(U\), that value cannot be in \(U\). Therefore \(f\) cannot choose an item from \(U\), so a fortiori \(f\) cannot be a choice function on \(P\). This argument shows that collections of sets of atoms do not necessarily have to have functions of choice, but fails to establish the same fact for ordinary math sets, for example the set of real numbers. This had to wait until 1963, when Paul Cohen showed that it was consistent with the standard axioms of the theory of sets (which prevent the existence of atoms) to assume that a numberable collection of pairs of actual number sets fails to have a function of choice. The core of Cohen's test method – the famous method of forcing – was much more general than any previous technique; however, the proof of its independence also made essential use of permutation and symmetry, essentially in the form in which Fraenkel originally engaged them. Gödel's evidence of the relative consistency of the AC with the theory of the set (see entry on Kurt Gödel) is based on a completely different idea: that of definability. He introduced a new hierarchy of sets — the constructive hierarchy — by analogy with the cumulative cumulative type Recall that the latter is defined by the following ordinal appeal, where \(\sP(X)\) is the power set of \(X\), \(\alpha\) is an ordinal, and \(\lambda\) is a limit ordinal:: \begin{align} V_0 &= \varnothing [V_{\alpha+1} \sP(V_{\alpha}) \\ V_{\lambda} &= {\bigcup}_{\alpha \lt \lambda} V_{\alpha} \{align} Constructable hierarchy is defined by a , where \(\Def(X)\) is the set of all subsets of \(X\) that are first-order definitions in the \(X, \in, (x)_{x\in X})\):[7] \begin{align} L_0 &= \varnothing [L_{\alpha+1} &= \Def(L_{\alpha}) \\ L_{\lamb i da} &= {\bigcup}_{\alpha\lt \lambda} L_{\alpha} \end{align} The constructable universe is class \(L = \bigcup_{\alpha\in \Ord} L_{\alpha}\); \(L\) members are buildable sets. Gödel pointed out that (assuming the axioms of the Zermelo-Fraenkel ZF set theory) the structure \((L, \in)\) is a model of ZF and also of AC, as well as the generalized continuum hypothesis). The relative consistency of AC with ZF follows. Gödel (1964) (and, independently, by Myhill and Scott 1971, Takeuti 1963 and Post 1951) noted that a simpler proof of the relative coherence of the AC can be formulated in terms of ordinal definition. If we write \(\rD(X)\) for the set of all subsets of \(X\) that are first-order defineable in the \((X, \in)\) structure, then the OD of the class of the ordinal defined sets is defined as the \(\bigcup_{\alpha\in \Ord} D(V_{\alpha})\). The HOD class of eredition ordinal definable sets consists of all sets for which \(a\), members \(a\), members \(a\), members \(a\), ... etc., all are ordinal definable. It can then be demonstrated that the structure (HOD, \(\in\ ) is a model of ZF + AC , from which the relative consistency of AC with ZF again follows. [8] 3. The maximum principles and Zorn's Lemma The Axiom of Choice is closely allied with a group of mathematical proposals collectively known as maximum principles. Broadly speaking, these proposals state that certain conditions are sufficient to ensure that a partially ordered set contains at least one maximum element, namely an element which, as regards the given partial order, no element strictly exceeds it. To see the link between the idea of a maximum element and AC, let's go back to formulating ac2 in terms of indexed sets. Consequently, the suppose we are given an indexed family of unempty sets \(\sA = \{A_{i}:i \in I\}.\) Let's define a potential choice function on \(\sA\) to be a \(f\) function whose domain is a subset of \(I\) so that \(f(i) \in A_{i}\) for all \(i\in J\). (Here the use of skill potential is suggested by the fact that the domain is a subset of \(I\); recall that a choice function \(f\) on \(\sA\) has the same properties as what we now call potential choice functions, except that the \(f\) domain is be all of \(I\), not just a subset.) Potential potential set \(P\) functions on \(\sA\) can be partially ordered by inclusion: we agree that for potential choice functions \(f, g \in P\), the relationship \(f \le g\) holds on condition that the \(f\) domain is included in that of \(g\) and the \(f\) value to an item in its domain coincides with the \(g\) value there. It is now easy to see that the maximum elements of \(P\) in terms of partial ordering \(\le\) are exactly the functions of choice on \(\sA\). Zorn's Lemma is the best known principle that ensures the existence of such maximum elements. To specify it, we need a few definitions. Given a partially ordered set \((P, \le)\), an upper limit for a subset \(X\) of \(P\) is an element \(a\in P\) for which \(x\le a\) for each \(x\in X\); a maximum element of \(P\) can then be defined as an item \(a\) for which the set of upper limits of \(\{a\}\) coincides with \(\{a\}\), which essentially means that no element in \(P\) is strictly greater than \(a\). A chain in \((P,\le)\) is a subset of \(C\) of \(P\) so that, for any \(x\), \(y\in P\), either \(x\le y\) or \(y\le x\). \(P\) is declared to be inductive if each chain in \(P\) has an upper limit. Now, Zorn's Lemma states: Zorn's Lemma (ZL): Each partially ordered unempty inductive inductive set has a maximum element. Why is Zorn's Lemma plausible? Here is an informal argument. Based on a partially unempty ordered inductive set \((P,\le)\), choose an arbitrary \(p_{0}\) item from \(P\). If \(p_{0}\) is maximum, stop here. Otherwise, choose a \(p_{1} \gt p_{0}\ item); if \(p_{1}\) is maximum, stop here. Otherwise, choose a \(p_{2} \gt p_{1}\) item and repeat the process. If none of the \(p_{0} \lt p_{1} \lt p_{2} \lt \ldots \lt p_{n} \lt \ldots\) is maximum, \(p_{i}\) forms a chain that, because \(P\) is inductive, has a higher link \(q_{0}\). If \(q_{0}\) is maximum, stop here. Otherwise, the procedure can be repeated with \(q_{0} \lt q_{1}\), ..., and then iterated. This process must eventually cease, because otherwise the joining of the chains thus generated would constitute an appropriate class, making itself an appropriate class contrary to the hypothesis. The point at which the process ends produces a maximum \(P\) item. This argument, duly rigorized, provides evidence[9] of the AC1 ZL in the Zermelo-Fraenkel set theory: in this ac1 evidence it is used to choose the elements mentioned in the informal argument. Another version of Zorn's Lemme can be given in terms of collections of sets. Given a \(\sH\) collection of sets, let's name a nest in \(\sH\) any subcollection \(\sN\) of \(\sH\) so that for any pair of members of \(\sN\), one is included in the other. [10] Call \(\sH\) strongly inductive if the union of any nest in \(\sH\) is a member of \(\sH\). Lemme Zorn can be reformulated equivalent as the claim that any strongly inductive nonempty collection \(\sH\) sets has a maximum member, that is, a member correspondingly included in any member of \(\sH\). This can, in turn, be formulated in a Call a family of powerfully reductive sets if it is closed under nesting intersections. Then any strong nonempty family of reductive sets has a minimal element, that is, a properly member, including no family member. AC2 is now easily derived from Lemme Zorn in this alternative form. For the \(P\) set of potential choice functions on an indexed family of sets \(\sA\) is clearly unempty and easily turns out to be highly inductive; thus Zorn's blade produces the existence of a choice function on \(\sA\). can be derived from ZL in a combined echo justification manner outlined above. Consequently, suppose that we are given a family \(\sH\) of mutually unempty disjunction sets; call a subset \(S \subseteq \bigcup \sH\) a sampling for \(\sH\) if, for any \(X\in \sH\), either \(X \subseteq S\) or \(S \cap X\) is unempletty and finite. Minimum sampling is exactly transverse for \(\sH\); [11] and the \(\sT\) collection of samplingists is clearly unempty because it contains \(\bigcup \sH\). So, if it can be shown that \(\sT\) is highly reductive,[12] Zorn's Lemma will produce a minimum element of \(\sT\) and thus a transverse for \(\sH\). The strong reduction of \(\sT\) can be seen as follows: suppose that \(\{S_{i}: i\in I\}\) is a sampling nest; allow \(S = {\bigcap}_{i\in I} S_{i}\). We must show that \(S\) is itself a sampling; for this purpose, leave \(X \in \sH\) and assume \(\eg(X \subseteq S)\). Then there is \(i \in I\) for which \(\eg(X \subseteq S_{i})\); because \(S_{i}\) is a sampling, \(S_{i}\cap X\) is finite and unempty, say \(S_{i}\cap X = \{x_{1}, \ldots, x_{n}\}\). Clearly \(S\cap X\) is then finished; suppose, for the sake of contradiction, that \(S\cap X = \varnothing\). Then, for each \(k = 1,\ldots,n\) there is \(i_k \in I\) for which \(\eg(x_k \in S_{i_k})\). It follows that \(\eg(S_{i} \subseteq S_{i_k})\), for \(k = 1, \ldots, n\). Thus, because \(S_{i}\) forms a chain, each \(S_{i_k}\) is a subset of \(S_{i}\). To \(S_{j}\) be the smallest of \(S_{i_1},\ldots, S_{i_k}\); then \(S_{j}\subseteq S_{i}\). But since \(\eg(x_{k} \in S_{j})\), for \(k = 1, \ldots, n\), it now follows that \(S_{j} \cap X = \varnothing\), contradicting the fact that \(S_{j}\) is a sampling. Therefore \(S \cap Xe \varnothing\); and \(S\) is a sampling so supported. We note that while Zorn's lem and the Axiom of Choice are theoretically equivalent, it is much more difficult to obtain the first of the latter than the other way around. Here's a brief timeline of maximum principles. 1909 Hausdorff introduces the first explicit formulation of a maximum principle and a derivative of AC (Hausdorff 1909) 1914 Grundzüge der Mengenlehre de Hausdorff (one of the first books on set theory and general topology) includes several maximal principles. 1922 Kunatowski formulates uses several maximum principles to avoid the use of transfinite ordinals (Kulatowski 1922). 1926–1928 Bochner and Others independently introducemaximum principles (Bochner 1928, Moore 1932). 1932). Max Zorn, apparently unfamiliar with the previous formulations of the maximum principles, publishes (Zorn 1935) his definitive version later to become celebrated as the lemme (ZL). ZL was first formulated in Hamburg in 1933, where Chevalley and Artin quickly adopted it. It seems that it was artin who first recognized that ZL would give AC, so the two are equivalent (over the remaining axioms of set theory). Zorn viewed his principle less as a theorem than an axiom—he hoped to replace cumbersome applications in the algebra of transfinite induction and well-orderedness, which the algebraists of the Noether school came to consider transcendental devices. 1939–1940 Teichmüller, Bourbaki and Tukey independently reformulate ZL in terms of finite properties (Bourbaki 1939, Teichmuller 1939, Tukey 1940). 4. Mathematical applications of the Axiom of Choice Has numerous applications in mathematics, a number of which have been officially equivalent to it[13]. Historically, the most important application was the first, namely: The well-ordered theorem (Zermelo 1904, 1908). Each set can be well ordered. After Zermelo published his 1904 proof of the AC ordering theorem, it was quickly seen that the two were equivalent. Another early equivalent of AC is the multiplier axiom (Russell 1906). The product of any set of non-zero cardinal numbers is non-zero. The first applications of AC include: Each infinite set has a subset named. This principle, again weaker than AC, cannot be proven without it in the context of the remaining axioms of the set theory. Each infinite cardinal number is equal to its square. This proved equivalent to AC in Tarski 1924. Each vector space has a base (initiated by Hamel 1905). This proved equivalent to AC in Blass 1984. Each field has an algebraic closure (Steinitz 1910). This claim is weaker than AC, indeed it is a consequence of the (weaker) compactness theorem for first-order logic (see below). There is an immeasurable Lebesgue set of real numbers (Vitali 1905). This later proved to be a consequence of bpI (see below) and therefore weaker than AC. Solovay (1970) established its independence from the remaining axioms of the set theory. A significant folklore equivalent of AC is the Law of Theoretical Distribution. For an arbitrary family of sets \({M_{i,j}: i \in I,j \in J}\), and where \(J^I\) is the set of all functions with domain \(I\) that take values in \(J\): \[\bigcap_{i\in I} \bigcup_{j\in J} M_{i,j} = \bigcup_{f\in J^I} \bigcap_{i\in I} M_{i,f(i)} \] A much-studied special case of AC is the Principle of Dependent Elections (Bernays 1942 , Tarski 1948). For any nonempty \(R\) relationship in a \(A\) set for which \(range(R) \subseteq \domain(R)\), there is a \(f: \omega \rightarrow A\) function so that, for \(n\in \omega, R(f(n),f(n+1))\). This principle, although (much) weaker than the AC, cannot be proven without the context of the remaining axioms of the set theory. Mathematical equivalents of AC include: Tychonov's Theorem (1930): the product of compact topological spaces is compact. This proved equivalent to AC in Kelley 1950. But for hausdorff compact spaces it is equivalent to BPI (see below) and, therefore weaker than AC Löwenheim-Skolem-Tarski Theorem (Löwenheim 1915, Skolem 1920, Tarski and Vaught 1957): a first-order sentence having a model of infinite cardinality \(\kappa\) also has a model of any infinite cardinality \(\mu\) so that \(\mu \le \kappa\). This has been proven equivalent to AC by Tarski. Krein-Milman theorem: the unit ball \(B\) of the dual of a real normative linear space has an extreme point, i.e. one that is not an inner point of any line segment in \(B\). This was found to be equivalent to AC in Bell and Fremlin 1972a. It shows that, given any indexed family \(\sA\) of nonempty sets, there is a natural bijection between the functions of choice on \(\sA\) and the extreme points of the unit ball of the dual of a certain normal real linear space constructed from \(\sA\). Each distributive lattice has a maximum ideal. This proved equivalent to AC in Klimovsky 1958, and for lattice sets in Bell and Fremlin 1972. Each identity switch ring has a maximum ideal. This was proven equivalent to the AC of Hodges 1979. There are a number of mathematical consequences of AC that are known to be weaker[14] than it is, in particular: the ideal prime Boolean theorem (BPI): each Boolean algebra has a maximum (or prime) ideal. This turned out to be weaker than AC in Halpern and Levy 1971. The theorem of stone representation for Boolean algebras (Stone 1936): each Boolean algebra is isomorphic in a set field. This is equivalent to BPI and therefore weaker than the AC Compactity Theorem for the logic of the first order (Gödel 1930, Malchev 1937, others): if each finite subset of a set of first-order sentences has a pattern, then the set has a pattern. This was shown, in Henkin 1954, to be equivalent to BPI, and therefore weaker than AC. The theorem of completeness for the first logical order (Gödel 1930, Henkin 1954): each consistent set of first-order sentences has a pattern. This was demonstrated by Henkin in 1954 to be equivalent to BPI, and therefore weaker than AC. If the cardinality of the model is correctly specified, the claim becomes equivalent to AC. Finally, there is sikorski extension theorem for Boolean algebra (Sikorski 1949): each complete Boolean algebra is injective, i.e. for any Boolean algebra \(A\) and any complete Boolean algebra \(B\) , any morphism of a subalgebra of \(A\) in \(B\) can be extended to the entire \(A\). The issue of equivalence of this with THE AC is one of the few questions remaining in this area; although it clearly involves BPI, it proved independent of BPI in Bell 1983. Many of these theorems are discussed in Bell and Machover (1977). 5. Axiomofa of choice and logic An initial connection between AC and occurs by reverting to its AC3 formulation in terms of relationships, namely: any binary relationship contains a function with the same domain. This version of AC is naturally expressable in a second-order language \(L\) with individual variables \(x\), \(y\), \(z\), ... and function variables \(f\), \(g\), \(h\), ... In \(L\), binary relationships are represented by formulas \(\phi(x, y)\) with two free individual variables \(x\), \(y\). The counterparty in \(L\) of AC3 statement is then ACL: \(\forall x \exists y \phi(x, y) \rightarrow \existes f \forall x \phi(x, fx).\) This sentence scheme is the standard logical form of the needle. Zermelo the original form of Axiom of Choice, AC1, can be expressed as a sentence scheme in an appropriate consolidated version of \(L\). Consequently, we now assume that \(L\) additionally contain predicated variables \(X\), \(Y\), \(Z\), ... and second-order function variables \(F\), \(G\), \(H\), ... Here you can apply a second-order function variable \(F\) to a predicated \(X\) variable to obtain an individual \(FX\). The AC1L sentence scheme: \(\forall X [\Phi(X) \rightarrow \existes x X(x)] \rightarrow \existes F \forall X[\Phi(X) \rightarrow X(FX)]\) is ac1's direct counterpart in this second-order consolidated language. In words, AC1L states that if each preached having a particular property \(\Phi\) has instances, then there is a function \(F\) on the predicates so that for any predicate \(X\) that satisfies \(\Phi\), \(FX\) is an instance of \(X\). Here the sermons play the role of sets. So far we have tacitly assumed our substantive logic to be ordinary classical logic. But the true depth of the connection between AC and logic occurs only when intuitionist or constructive logic is brought into the image. It is a remarkable fact that, assuming only the framework of intuitionist logic, together with certain slight assumptions below, the Axiom of Choice can be demonstrated as involving the cardinal rule of classical logic, the law of the excluded means — the assertion that \(A \vee eg A\) for any proposal \(A\). Specifically, using the rules of intuitionist logic in our augmented language \(L\), we will deduce[15] the law of the excluded means from AC1L related to the following additional principles: Predictive Understanding: \(\eximis X \forall x[X(x) \leftrightarrow \phi(x)])), where \(\phi\) contains no related or predicated functions. Function extension: \(\forall X \forall Y \forall F[X \approx Y \rightarrow FX = FY]\), where \(X \approx Y\) is an abbreviation for \(\forall x[X(x) \leftrightarrow Y(x)]\), i.e. \(X\) and \(Y\) are extensionally equivalent. Two distinct persons: \(\uO e \uI\), where \(\uO\) and \(\uI\) are individual constants. Now let \(A\) be a given proposal. Through Predictive Understanding, we can enter predicated constants \(U\), \(V\) along with the statements \begin{align} \tag{1} &\forall x[U(x) \leftrightarrow (A \vee = 0)] \\ otag otag x[V(x) \leftrightarrow (A \vee x = 1)] \end{align} Let \(\Phi(X)\) to be the formula \(X \aprot U \vee X \aprot V\). Then we can clearly state \(\forall X[\Phi(X) \rightarrow \exists xX(x)]\) so that AC1L can be invoked to state \(\exist F \forall X[\Phi(X) \rightarrow X(FX)]\). We can now enter a constant \(K\) function along with the statement \[\tag{2} \forall X[\Phi(X) \rightarrow X(KX)]. \] Since we can obviously state \(\Phi(U)\) and \(\Phi(V)\), it follows from (2) that we can state \(U(KU)\) and \(V(KV)\), also using (1), \[[A \vee KU = 0] \wedge [A \vee KV = 1]. \] Use of the law (which holds in intuitionist logic) , it follows that we can state \[A \vee [KU = 0 \wedge KV = 1]. \] From the assumption that \(0 e 1\) it follows that \[\tag{3} A \vee KU e KV \] is assertive. But it follows from (1) that we can state \(A \rightarrow U ≈ V\), and so also using the function extension, \(A \rightarrow KU = KV\). This produces the assertiveness \(KU e KV \rightarrow e.g. A\), which, along with (3) in turn, yields the assertiveness \[A \vee for example A \] that is, the excluded middle law. The fact that the Axiom of Choice implies the Exclusion of the Middle seems at first glance to contradict the fact that the former is often taken as a valid principle in the systems of constructive mathematics governed by intuitionist logic, for example the Constructive Analysis of the Bishop[16] and Martin-Löf's Constructive Theory[17], in which the excluded Middle East is not affirmed. In Bishop's words, A function of choice exists in constructive mathematics, because a choice is involved even by the very meaning of existence. Thus, for example, the antecedent \(\forall x \exists y \phi(x,y)\) in the ACL, given a constructive interpretation, only means that we have a procedure that, applied to each \(x\), produces a \(y\) for which \(\phi(x, y)\). But this is exactly what is expressed accordingly \(\there is f \forall x \phi(x,fx)\) in ACL. In order to solve the difficulty, we note that the derivation of Excluded Middle from ACL1 used the principles of predictive understanding and extension of functions. It follows that, in systems of constructive mathematics which affirm the AC (but not excluded by the middle), either the principle of predictive understanding or the principle of extension of functions must fail. While the principle of predictive understanding can be given a constructive justification, no such justification can be provided for the principle of extension of functions. The functions on the predicates are given intensely, and satisfy only the corresponding principle of intensity \(\forall X \forall Y \forall F[X = Y \rightarrow FX = FY]\). The principle of extensiality can easily be made to fail by taking into account, for example, the predicates \(P\): biped rationally without feathers and \(Q\): human being and the function \(K\) on the predicates that assign each predicate the number of words in the description Then we can. Can. that \(P \approx Q\) but \(KP = 3\) and \(KQ = 2\). In the theory of the intuitionist set (i.e. the theory of the intuitionist set, as opposed to classical logic – we abbreviate this as ANT) and in the topos theory the principles of predictive understanding and extension of functions (both properly interpreted) hold and thus the needle implies the exclusion of the means. [19] , [20] The derivation of the means excluded from the AC was first given by Diaconescu (1975) in a theoretical category framework. His proof used essentially different ideas from the evidence presented above; in particular, it does not use the principles of enlargement, but uses the idea of the coefficient of an object (or set) through a relationship of equivalence. It is instructive to formulate Diaconescu's argument in the IST. To do this, let's name a \(U\) subset of a detachable \(A\) set if there is a \(V\) subset of \(A\) for which \(U \cap V = \varnothing\) and \(U \cup V = A\). Diaconescu's argument amounts to a derivation from AC4 (see above) of the claim that each subset of a set is detachable, from which The Middle is easily excluded. There he is. First, given \(U \subseteq A\), an indicator for \(U\) (in \(A\)) is a map \(g: A \times 2 \rightarrow 2\) that satisfies \[U = \{ x \in A: g(x,0) = g(x,1) \} \] It is then easy to show that a subset is removable if and only if it has an indicator. Now we will show that if AC4 holds, then any subset of a set has an indicator, and therefore is removable. For \(U \subseteq A\), let \(R\) be the binary relationship on \(A + A = A \times \{0\} \cup A \times \{1\}\) date of \[R = \{ ((x,0),(x,0)): x \in A \} \cup \{ ((((((((((x,0),(x,0)): x \in A \\ } \cup \{ ((((((((((((x,0),(x,0)): x \in A \} \{((((((((((((((((((x,0),(x,0): x \in \ {((,0),(x,0)): x \in A \} \cup \{ (((((((x,x,1),(x,1)): x \in A \} \cup \{ ((x,0),(x,1) : x \in A \} \cup \{ ((x,1),(x,0) \} : x \in A \] You can verify that \(R\) is an equivalence relationship. Write \(r\) for the natural map from \(A + A\) to the coefficient[21] \(Q\) of \(A + A)\) of \(R\) that transports each member of \(A + A\) to its \(R-\)equivalence class. Now apply AC4 to get a map \(f: Q \rightarrow A + A\) that satisfies \(f(X) \in X\) for all \(X \in Q\). It is then not difficult to show that by writing \(\pi_{1}\) for projection on the first coordinate, \[\tag{*} \mbox{for } n=0.1 \mbox{ and } x \in A, \pi_1(f(r(x,n)))) = x; \] and \[\tag{**} x \in U \leftrightarrow f(r(x,0)) = f(r(x,1)). \] Now defined \(g: A \times 2 \rightarrow 2\) by \(g = \pi_{2} \circ f \circ r\) , where \(\pi_2\) is a projection on the second coordinate. Then \(g\) is an indicator for \(U\), showing the following equivalences: \begin{align} x \in U &\leftrightarrow f(r(x,0))=f(r(x,1)) \ldots \mbox{by (**)} \\ &\leftrightarrow \pi_{1}().f(r(x.0)) = \wedge \pi_{2}(f(r(x,0))) = \pi_{2}(f(r(x,1))) \\ &\leftrightarrow \pi_{2}(f(r(x ,0))) = \pi_{2}(f(r(x,1))) \ldots \mbox{using (*)} \\ &\leftrightarrow g(x.0) = g(x,1). \end{align} Proof is complete. It can be demonstrated (Bell 2006) that each of several intuitively invalid logical principles, including the law of (in the theory of the intuitionist set) to a properly weakened version of the axiom of choice. Consequently, these logical principles can be considered as principles of choice. Here are the logical principles in question: SLEM \(\alpha \vee eg \alpha\) (\(\alpha\) any sentence) Lin \((\alpha \rightarrow \beta) \vee (\beta \rightarrow \\ alpha)\) (\(\alpha\), \(\beta\) any sentences) Stone \(for example \alpha \vee eg \alpha\) (\(\alpha\) any sentence) Ex \(\exists x\ \exist x \alpha(x) \rightarrow \alpha(x)]\) (\(\alpha(x)\) any formula with no more than \(x\) free) A \(\exime x(\alpha(x) \right arrow \forall x\alpha(x)]\) (\(\alpha(x)\) any formula with no more than \(x\) free) Dis \(\forall x\alfa \evere\beta) x)] \ rightarrow \alpha\vee\forall x\beta(x)\) (\(\alpha\) any sentence, \(\beta(x)\) any formula with no more \(x\) free) Over intuitionist logic, Lin, Stone and Ex are consequences of SLEM; and A implies Dis. All these schemes follow, of course, from the complete law of the excluded means, i.e. SLEM for arbitrary formulas. In the following, the empty set is denoted by 0, \({0}\) with 1 and \({0, 1}\) by 2. We formulate the following principles of choice — here \(X\) is an arbitrary set, \(Fun(X)\) function class with domain \(X\) and \(\phi(x,y)\) an arbitrary formula of the language of the set theory with no more free variables \(x\), \(y\)): AC\(_X\) \(\forall x\in X\ \exist y \phi(x,y) \rightarrow \exist f \in \Fun(X)\ \forall x \in X\ \phi(x ,fx)\) AC\(^*_X\) \({\exist f \inn \Fun(X)} [{\forall x\inn X}\ \exist y \phi(x,y) \rightarrow {\forall x \inn X}\ \phi(x,fx)]\) DAC\(_X\) \\\forall f \inn Fun(X)\ {\exist x \inn X}\ \phi(x,fx) \rightarrow {\exists x \inn X}\ \forall y \phi(x,y)\) DAC(^*_X\) \({\exist f \inn \Fun(X)}\ [{\exist x\inn X}\ \phi(x ,fx) \rightarrow {\exists x \inn X}\ \forall y \phi(x,y)]\) The first two of these are AC forms for \(X\); while the classic equivalent in IST AC\(^*_X\) involves AC\(_X\), but not the other way around. The principles of DAC\(_X\) and DAC(^*_X\) are double forms of the axiom of choice for \(X\): classically both are equivalent to AC\(_X\) and AC\(^*_X\), but intuitiveLY DAC\(^*_X\) implies DAC\(_X\), and not vice versa. We also formulate the weak principle of extensional selection, where \(\alpha(x)\) and \(\beta(x)\) are formulas with a free variable at most: WESP: \({\exime x\inn 2}\ \alpha(x) \wedge {\exime x \inn 2}\ \beta(x) \rightarrow {\\\\{\\\\ {\\\exists x \inn 2}\ {\exists y \inn 2}\ [\alpha(x) \wedge \beta(y) \wedge [{\forall x \inn 2}\ [\alpha(x) \leftrightarrow \beta(x)] \rightarrow x = y]].\ , a simple consequence of the axiom of choice, states that for any pair of instantiated properties of members of 2, the instances can be assigned to the properties in a way that depends only on their extensions. Each of the logical principles tabulated above is equivalent (in IST) to a principle of choice. In fact: WESP and SLEM are equivalent to IST. AC\(^*_1\) and Ex are equivalent Moreover, while DAC\(_1\) is easy to see as demonstrable in IST, we have DAC\(^*__1\) and Un are equivalent over IST. Then, while AC \(_2\) is easy to prove in IST, by contrast we have DAC \(_2\) and Dis are equivalent over IST. Over IST, DAC\(^*_2\) is equivalent to One and therefore also TO DAC\(^*_1\). To provide Lin and Stone equivalent choice schemes we enter ac\(^*_X\): \({\exist f \inn 2^X} [{\forall x\inn X}\ {\exist s y \inn 2}\ \phi(x,y) \rightarrow {\exists x \inn X}\ \2 phi(x,fx)]\) wac\(^*_X\): \({\exist f \inn 2^X} [{\forall x \inn X}\ {\exist y \inn 2}\ \phi(x,y) \rightarrow {\forall x \inn X}\ \phi(x ,fx)]\) provided it is demonstrable in IST that \(\forall x[\phi(x,0) \rightarrow eg\phi(x,1)]\) Clearly ac\(^*_X\) is equivalent to \({\exists f \inn 2^X}[{\forall x \inn X}[\phi(x,0) \vee \phi(x,1)] \rightarrow {\forall x \inn X}\ \phi(x,fx)]\) and similar for if\(^*_X\). Then, over IST, act(^*_1\) and if(^*_1\) are equivalent, respectively, to Lin and Stone. These results show how deeply the principles of choice interact with logic when background logic is supposed to be intuitive. In a classic setting in which the excluded Middle Law is assumed that these connections are deleted. Readers interested in the theme of the axiom of choice and type theory can consult the following additional document: Axioma of choice and type theory